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On the Derivation of the Incompressible Navier–Stokes Equation for Hamiltonian Particle Systems

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We consider a Hamiltonian particle system interacting by means of a pair potential. We look at the behavior of the system on a space scale of order ε^{-1}, times of order ε^{-2} and mean velocities of order ε, with ε a scale parameter. Assuming that the phase space density of the particles is given by a series in ε (the analog of the Chapman–Enskog expansion), the behavior of the system under this rescaling is described, to the lowest order in ε, by the incompressible Navier–Stokes equations. The viscosity is given in terms of microscopic correlations, and its expression agrees with the Green–Kubo formula.

KEY WORDS: Hydrodynamic limit; incompressible Navier–Stokes equations; particle systems.

1. INTRODUCTION

A system of many interacting particles moving according to the Newton equations of motion can be described on a space scale much larger than the typical microscopic scale (say, the range of the interaction) in terms of density, velocity, and temperature fields, satisfying hydrodynamic equations such as Euler or Navier–Stokes equations. The scale separation and the local conservation laws are responsible for this reduced description. In fact, on the macroscopic scale the quantities which are locally conserved (slow modes) play a major role in the motion of the fluid. The derivation of the Euler equations is based on the assumption of local equilibrium. On times of order ε^{-1}, the system is expected to be described approximately by a

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local Gibbs measure, with parameters varying on regions of order ε^{-1}, ε being a scale parameter. The local equilibrium assumption implies that the parameters of the local Gibbs measure satisfy the Euler equations.[1,2] The microscopic structure (the potential) appears only in the state equation which links pressure and internal energy to the other macroscopic parameters. The microscopic locally conserved quantities converge, as $\varepsilon \to 0$, by a law of large numbers, to macroscopic fields. To make this correct, the many-particle Hamiltonian system must have good dynamical mixing properties to approach and stay in a state close to the local equilibrium. At the moment it is not understood how to provide such properties. Therefore the only rigorous results are obtained by adding some noise to the Hamiltonian evolution[3] (see ref. 4 for a review on the rigorous results for stochastic systems).

The situation is very different for the derivation of the Navier–Stokes (NS) equations. These equations, which describe the behavior of a fluid in the presence of dissipative effects, do not have an immediate interpretation in terms of scale separation. This is not surprising because the NS equations do not have a natural space-time scale invariance like the Euler equations. In fact, to see the effect of the viscosity and the thermal conduction one has to look at times such that neighboring regions in local equilibrium exchange a sensible amount of momentum and energy. Simple considerations show that the right scale of time is ε^{-2}. On the other hand, we cannot hope to find the Navier–Stokes behavior under the rescaling $x \to \varepsilon^{-1}x$ and $t \to \varepsilon^{-2}t$ since the NS equations are not invariant under this scaling, due to the presence of the transport terms. Therefore we consider the incompressible limit simultaneously, because the incompressible Navier–Stokes equations (INS) have the required scaling invariance.

To explain this point let us recall the derivation of the hydrodynamic equations from the Boltzmann equation, which describes the large-scale dynamics of a gas in the low-density or kinetic regime. In this regime the typical scales are the mean free path and the mean free time, and every particle undergoes collisions only once in a while. To recover the hydrodynamic behavior one has to look at the system on space and time scales which are very long with respect to the mean free path and the mean free time, in such a way that every particle can have so many collisions that in the macroscopic time it has thermalized. To be precise, it has been proved in ref. 5 that, if we rescale both space and time by ε^{-1}, the solution of the rescaled Boltzmann equation looks like a Maxwellian with parameters solving the Euler equations, for small ε. Now ε is the scale separation parameter between the mean free path and the typical macroscopic scale. To get sensible viscous effects, the time has to be of order ε^{-1} compared to the Euler times, hence one has to consider the parabolic space-time scaling ($x = \varepsilon^{-1}x'$, $t = \varepsilon^{-2}t'$). On this time scale one makes the transport term finite by taking the Mach number[6] $Ma = U/c$ (where U is a typical velocity and c is the sound speed) of order ε. This corresponds to the incompressible regime. In ref. 7 (see also ref. 8 for the nonsmooth case), it is proved that, if $u(x, t)$ is a sufficiently smooth solution of the incompressible Navier–Stokes equations on a torus for $t \in [0, t_0]$, one can construct a solution f^ε to the rescaled (parabolically) Boltzmann equation such that, for $t \in [0, t_0]$,

$$\| f^\varepsilon - M(\rho, \varepsilon u, T) \|_\infty < c\varepsilon^2 \tag{1.1}$$

where ρ and T are given positive constants.

In the incompressible regime the macroscopic state is described by a divergenceless velocity field $u(x, t)$, constant density ρ, and constant temperature T. The pressure $p(x, t)$ appearing in the equations is no longer related to the thermodynamic parameters by means of a state equation, but has simply the meaning of a Lagrangian multiplier for the constraint $\operatorname{div} u = 0$. The INS equations are

$$\operatorname{div} u = 0 \tag{1.2}$$

$$\rho \frac{\partial u}{\partial t} + \rho u \cdot \nabla u - \eta \, \Delta u = -\nabla p \tag{1.3}$$

η is the viscosity and microscopic interactions enter only into its determination.

The above dimensional analysis of course can be carried out as well for a particle system. Along this path, in this paper we give a formal derivation of the INS from a Hamiltonian particle system under the parabolic rescaling, in the low-Mach-number regime.

The main ingredient is the assumption that the nonequilibrium density can be expressed as a truncated series in the parameter ε. We follow a procedure inspired by the Hilbert–Chapman–Enskog expansion used to construct the solution of the rescaled Boltzmann equation. From the physical point of view we think of the system as being in local equilibrium with parameters which are themselves given by a series in ε. However, there is a nonhydrodynamic correction to the local equilibrium which depends on the nonconserved quantities in the system (fast modes) and we assume that this correction does not affect the first order in the expansion, that is, at the first order the system is still described by a local Gibbs measure with parameters which differ from constants by terms of order ε. This is strictly related to the incompressibility assumption and would be false in the case of finite Mach number. This assumption is the translation of (1.1) to the particle system case. On the other hand, the nonhydrodynamic corrections in the second order are important on the scale ε^{-2} and give rise to the NS terms. It is worth to mention here that very strong and rather uncontrollable assumptions are necessary even to give sense to the formal calculations below:

(i) The space of the invariant observables for the microscopic dynamics reduces to the locally conserved quantities, mass, momentum, and energy.

(ii) Some equilibrium time correlation functions decay sufficiently fast.

Such assumptions are far from being sufficient for a mathematical proof. Actually, in the only case for which we have a rigorous proof (see Section 4), we use much more on the generator of the dynamics, such as the spectral gap and logarithmic Sobolev inequalities. Most of the argument below is based on the assumption that the inverse of the generator makes sense at least on some suitable set of observables.

Under this assumptions it is possible to determine the form of the lowest nonhydrodynamic correction and thus conclude that the conservation equations are well approximated by the INS for ε small.

We only examine the expected values of the empirical fields and do not look for a law of large numbers. The reason for this choice is that, as explained below, the quantity

$$\varepsilon^{d-1} \sum_i v_i \delta(x_i - x) \tag{1.4}$$

is the candidate for approximating the mean velocity field. In dimensions 1 and 2 the fluctuations of momentum are of the same order as or even bigger than this quantity. Hence it cannot converge to a deterministic field and a law of large numbers is not true. The situation is different in higher dimension, but we confine ourselves to the analysis of the averages.

Our derivation gives the viscosity coefficient η in terms of a global equilibrium time correlation function

$$\eta = \frac{1}{2T} \int_0^\infty d\tau \int d\xi \, \langle \bar{w}^{12}(\xi, \tau)\, \bar{w}^{12}(0, 0) \rangle \tag{1.5}$$

where $\bar{w}$ is the "modified" velocity current tensor defined in Section 3. The fluctuation-dissipation theorem relates the transport coefficients to time-integrated correlation functions.[9,10] The expression we find agrees with the Green–Kubo formula for the viscosity. We also obtain the Green–Kubo formula for the bulk viscosity ζ, but as is well known, it does not appear in the INS. In our case the viscosity is not a function of space and time, since the correlation is evaluated at the global equilibrium, as a consequence of the fact that we are studying the incompressible regime, where density and temperature are constant. As a by-product of our analysis, we find the analog of the Boussinesq condition for the first correction P_1 of the thermodynamic pressure P_ε. In fact we have

$$\nabla P_1 = 0 \tag{1.6}$$

Moreover, the conservation law for the energy gives, in the incompressible approximation, an equation for the first correction to the temperature T_1,

$$c(\partial_t T_1 + u \cdot T_1) = \kappa \, \Delta T_1 \tag{1.7}$$

where κ is the conductivity, which is given again in terms of an integrated equilibrium time correlation, and c is the specific heat at constant pressure. Our approach to derive the Green–Kubo formulas seems similar to the one followed by Green.[11] It is an alternative way to obtain them with respect to the linear response theory of Green and Kubo. A different derivation is due to Zubarev,[12] which proposes a specific form of the nonequilibrium distribution allowing one to derive the expressions of the transport coefficients as well as the Navier–Stokes corrections.[13]

Finally we remark again that, to try to prove the hydrodynamic limit under the above rescaling, we need ergodic properties of the Hamiltonian system even stronger than the ones needed for the Euler regime. Of course they are beyond the available mathematical techniques. It would be very interesting to prove the analog of the result of ref. 3 in this case, i.e., that adding a suitable noise to the dynamics, the hydrodynamic limit is achieved. Even this is far from being at hand, because there are many technical difficulties in dealing with nongradient systems in the continuous space.

It is therefore natural to try to explore this setting in some simple lattice models with stochastic dynamics which have all the required ergodic properties. This program has been successfully accomplished in ref. 14 for the asymmetric simple exclusion process (ASEP) (see refs. 4 and 15 for references) in dimension bigger than 2. In fact, the ASEP is probably the simplest nontrivial model for which an analog of the incompressible limit makes sense. In this model there is only one conserved quantity, the density, and the analog of the Euler limit is well known.[16] The limiting equation for the density is the nonviscous Burgers equation. The diffusive scaling limit presents mostly the same difficulties (absence of scale invariance) discussed above. Moreover, from the technical point of view, the study of the model is made difficult by the fact that it is a nongradient system (in the sense that the current is not the "gradient" of some function; see ref. 4) and the dynamics does not satisfy the detailed balance with respect to the invariant product measure. In ref. 14 we consider an initial state which is a product measure with a density profile with spatial fluctuation of order ε (the analog of the assumption of mean velocities of order ε in Hamiltonian systems). Using the Varadhan method[17] of dealing with nongradient systems, the entropy method,[18,3] and a kind of multiscale analysis, it is possible to prove that, with probability 1, the fluctuation of the empirical density around the constant profile is of order ε and the rescaled fluctuation of density, after a suitable space shift of order ε^{-1}, satisfies the viscous Burgers equation in the limit $\varepsilon \to 0$. An important feature, to our purposes, is that the diffusion matrix computed in ref. 14 is strictly bigger than the one of the corresponding symmetric process. This means that the diffusivity of the model is not just that due to the assumed stochasticity of the model (the symmetric part), but there is a contribution from the "deterministic" motion of the particles (the asymmetric part), which can be interpreted as a "Navier–Stokes" contribution in analogy with the Hamiltonian case. This is in agreement with the heuristic Green–Kubo formula for this model, as computed in ref. 19.

2. THE CONSERVATION LAWS

We consider a system of N identical particles of unit mass in a cube of size ε^{-1} in $\mathbb{R}^d$, with periodic b.c., interacting through a pair central potential V of finite range. After rescaling space as ε^{-1} and time as ε^{-2} the Newton equations become, for $i = 1,..., N$,

$$\begin{aligned} \frac{dx_i}{dt}(t) &= \varepsilon^{-1} v_i(t) \\ \frac{dv_i}{dt}(t) &= -\varepsilon^{-2} \sum_{i \neq j} \nabla V(\varepsilon^{-1}(x_i - x_j)) \end{aligned} \tag{2.1}$$

The number of particles N is assumed to be of order ε^{-d} to keep the density finite. The total number of particles, the d components of the total momentum, and the total energy are the conserved quantities. We construct the corresponding empirical fields:

Empirical density

$$z^0(x) = \varepsilon^d \sum_i \delta(x_i - x) \tag{2.2}$$

Empirical velocity field density

$$z^{\alpha}(x)=\varepsilon^{d}\sum_{i} v_i^{\alpha}\delta(x_i-x), \qquad \alpha=1,\dots, d \tag{2.3}$$

Empirical energy density

$$z^{d+1}(x)=\varepsilon^{d}\sum_{i}\tfrac{1}{2}\left[v_i^2+\sum_{j\neq i}V(\varepsilon^{-1}|x_i-x_j|)\right]\delta(x_i-x) \tag{2.4}$$

Their meaning is as follows: The average of the integral of z^{α} over a small region is equal to the average number of particles, momentum, energy associated to the region. We will write also

$$z^{\mu}(x)=\varepsilon^{d}\sum_{i}\delta(x_i-x)\,z_i^{\mu} \tag{2.5}$$

with

$$z_i^0=1; \qquad z_i^{\alpha}=v_i^{\alpha}, \quad \alpha=1,\dots, d; \qquad z_i^{d+1}=\tfrac{1}{2}\left[v_i^2+\sum_{i\neq j}V(\varepsilon^{-1}|x_i-x_j|)\right]$$

The generalized functions z^{α} on the phase space are expected to be approximated, to the lowest order in ε, by the macroscopic hydrodynamic fields, in the sense that, with probability 1, for any smooth function f, we have

$$\int dx\, z(x)\, f(x)=\int dx\, b(x)\, f(x)+o(1) \tag{2.6}$$

where $o(1)$ denotes a quantity going to 0 as $\varepsilon\to 0$, $z=\{z^{\alpha}\}$, and $b=\{\rho, U, e\}$, the macroscopic density, velocity field, and energy, respectively. The empirical fields satisfy the following local conservation laws, which are obtained differentiating $z^{\alpha}(x,t)$ with respect to the time and using the Newton equations:

$$\frac{d}{dt}\varepsilon^{d}\sum_{i} f(x_i)=\varepsilon^{-1}\varepsilon^{d}\sum_{i}\frac{\partial f}{\partial x_i^{\alpha}}(x_i)\,v_i^{\alpha} \tag{2.7}$$

$$\frac{d}{dt}\varepsilon^{d}\sum_{i} f(x_i)\,v_i^{\beta}=\varepsilon^{-1}\varepsilon^{d}\sum_{i}\left\{\frac{\partial f}{\partial x_i^{\alpha}}(x_i)\,v_i^{\alpha}v_i^{\beta}\right.$$
$$\left.-\varepsilon^{-1}\sum_{j\neq i}\nabla_{\beta}V(\varepsilon^{-1}(x_i-x_j))\,f(x_i)\right\} \tag{2.8}$$

$$\frac{d}{dt}\varepsilon^d \sum_i f(x_i)\, z_i^{d+1} = \varepsilon^{-1}\varepsilon^d \sum_i \left\{ \frac{\partial f}{\partial x_i^\alpha}(x_i)\, v_i^\alpha z_i^{d+1} - \frac{1}{2}\varepsilon^{-1} \sum_{i \neq j} \nabla_\alpha V(\varepsilon^{-1}(x_i - x_j))\, v_i^\alpha f(x_i) \right\} \tag{2.9}$$

Here $\nabla_\beta V(\xi) = \partial V(\xi)/\partial \xi_\beta$. Because of the symmetry properties of the potential we can write, as usual, the second term in the r.h.s. of (2.8) as

$$-\tfrac{1}{2}\varepsilon^{d-2} \sum_{i \neq j} \nabla_\beta V(\varepsilon^{-1}(x_i - x_j))[f(x_i) - f(x_j)] \tag{2.10}$$

Since f is slowly varying on the microscopic scale, we can write, with $\xi_i = \varepsilon^{-1} x_i$,

$$\begin{aligned} f(\varepsilon\xi_i) - f(\varepsilon\xi_j) = {} & \sum_\gamma \frac{\partial f}{\partial x_i^\gamma}(x_i)\, \varepsilon[\xi_i^\gamma - \xi_j^\gamma] \\ & + \sum_{\gamma,\nu} \frac{\partial^2 f}{\partial x_i^\gamma\, \partial x_i^\nu}(x_i)\, \varepsilon^2 [\xi_i^\gamma - \xi_j^\gamma][\xi_i^\nu - \xi_j^\nu] \\ & + \varepsilon^3 D(x_i - x_j) + O(\varepsilon^4) \end{aligned} \tag{2.11}$$

where

$$D(x_i - x_j) = \sum_{\gamma,\nu,\alpha} \frac{\partial^3 f}{\partial x_i^\gamma\, \partial x_i^\nu\, \partial x_i^\alpha}(x_i)[\xi_i^\gamma - \xi_j^\gamma][\xi_i^\nu - \xi_j^\nu][\xi_i^\alpha - \xi_j^\alpha]$$

Due to the symmetry of the potential the second term of the Taylor expansion of f does not contribute and the last term of Eq. (2.8) becomes

$$\frac{1}{2}\varepsilon^{-1}\varepsilon^d \sum_{i,j} \sum_\gamma \frac{\partial f}{\partial x_i^\gamma}(x_i)\, \Psi^{\beta\gamma}(\varepsilon^{-1}(x_i - x_j)) + O(\varepsilon) \tag{2.12}$$

with

$$\Psi^{\beta\gamma}(\xi) = -\nabla_\beta V(\xi)\, \xi^\gamma \tag{2.13}$$

An analogous computation can be done for the energy equation. The general form of the rescaled local conservation laws is

$$\frac{\partial}{\partial t} \int dx\, f(x)\, z^\beta(x) = \varepsilon^{-1} \int dx \sum_{k=1}^{3} \frac{\partial f}{\partial x^k}\, w^{\beta k}(x) + O(\varepsilon) \tag{2.14}$$

where $w^{\beta k}$, $\beta = 0, ..., d+1$; $k = 1, ..., d$ are the currents associated to the fields z^β and are explicitly given by

$$w^{0k}(x) = \varepsilon^d \sum_i \delta(x_i - x)\, v_i^k \tag{2.15}$$

$$w^{\beta k}(x) = \varepsilon^d \sum_i \delta(x_i - x) \left\{ v_i^\beta v_i^k + \tfrac{1}{2} \sum_j \Psi^{\beta k}(\varepsilon^{-1}(x_i - x_j)) \right\}, \quad \beta = 1,..., d \tag{2.16}$$

$$w^{d+1,k}(x) = \varepsilon^d \sum_i \left\{ v_i^k z_i^{d+1} + \tfrac{1}{2} \sum_{j,\gamma} \Psi^{\gamma k}(\varepsilon^{-1}(x_i - x_j))\, \tfrac{1}{2}\, [v_i^\gamma + v_j^\gamma] \right\} \tag{2.17}$$

We put also $w^{\beta k}(x) = \varepsilon^d \sum_i \delta(x_i - x)\, w_i^{\beta k}$.

The empirical fields $z^\alpha(x)$ are approximate integrals of the motion in the sense that, defining the Liouville operator in terms of microscopic variables $\xi_i = \varepsilon^{-1} x_i$ as

$$\mathscr{L} f(\xi, v) = \sum_i \left\{ v_i^\alpha \frac{\partial f}{\partial \xi_i^\alpha} - \sum_{i \neq j} \frac{\partial V}{\partial \xi_i^\alpha} (|\xi_i - \xi_j|) \frac{\partial f}{\partial v_i^\alpha} \right\} \tag{2.18}$$

and denoting by ζ_i^α the quantities z_i^α as functions of the microscopic variables ξ_i, it follows from the previous calculation that

$$\mathscr{L} \left[\varepsilon^d \sum_i f(\varepsilon \xi_i)\, \zeta_i^\alpha \right] = O(\varepsilon) \tag{2.19}$$

We call the observables with this property *local integrals of motion.* This is consistent with the following definition of the *local equilibrium distribution* on the phase space (in microscopic variables):

$$G = Z^{-1} \exp \sum_i \sum_{\alpha=0}^{d+1} \lambda^\alpha(\varepsilon \xi_i)\, \zeta_i^\alpha \tag{2.20}$$

with Z the normalization factor. In fact the distribution G is locally stationary for $\mathscr{L}$ in the sense that

$$\mathscr{L} G = O(\varepsilon) \tag{2.21}$$

In other words, if we look at a region around the point x microscopically very large but macroscopically small, the system appears to be in equilibrium in this region and its distribution is the Gibbs measure G restricted to the variables localized there. In regions of this type we can follow the evolution of the system for very large microscopic times τ such that it makes sense to consider ergodic properties of the unitary group S_τ

generated by $\mathscr{L}$. For any local observable ϕ of mean zero with respect to G we put

$$\hat{\phi} = \lim_{\tau \to \infty} \tau^{-1} \int_0^\tau d\tau' \, S_{\tau'} \phi \tag{2.22}$$

$\hat{\phi}$ represents the part of ϕ which is invariant under S_τ.

We *assume* that the set of all the invariant local observables contains only combinations of the empirical fields associated to the particle number, momentum, and energy, and any function of them.

To make this concept more precise, we refer to refs. 20 and 4, which introduce the Hilbert space of the local observables equipped with the scalar product

$$(\phi, \psi) = \int dx \, [\langle \phi \tau_x \psi \rangle - \langle \phi \rangle \langle \psi \rangle] \tag{2.23}$$

Here $\langle \cdot \rangle$ is the average on the local Gibbs measure. Our assumption means that, introducing the projector on the invariant space defined as

$$\mathscr{P}\phi = \sum_{\mu=0}^{d+1} (\phi, z^\mu)(z, z)^{-1}_{\mu\nu} \, z^\nu \tag{2.24}$$

where $(z, z)^{-1}$ denotes the inverse of the matrix with elements (z_μ, z_ν), then

$$\hat{\phi} = \mathscr{P}\phi \tag{2.25}$$

To show how the conservation laws give the hydrodynamic equations (INS), we follow a procedure similar to the one Chapman and Enskog proposed to approximate the solutions of the Boltzmann equation. Let us start with the phase space distribution function F_ε for the rescaled system, which satisfies the Liouville equation

$$\frac{\partial F_\varepsilon}{\partial t} = \varepsilon^{-2} \mathscr{L}^* F_\varepsilon \tag{2.26}$$

where $\mathscr{L}^*$ is the adjoint, w.r.t. the Liouville measure, of the Liouville operator on the phase space, formally given by $\mathscr{L}^* = -\mathscr{L}$.

Writing F_ε as a series in ε, $F_\varepsilon = \sum_n \varepsilon^n F^n$, and substituting it in (2.26), we get the diverging terms $\varepsilon^{-2}\mathscr{L}^* F_0$ and $\varepsilon^{-1}\mathscr{L}^* F_1$. Therefore we are forced to put $\mathscr{L}^* F_0 = 0$, hence F_0 has to be the global equilibrium. Moreover, $\varepsilon^{-1}\mathscr{L}^* F_1$ is finite if $\mathscr{L}^* F_1 = O(\varepsilon)$. This means that the term of order ε has to be a function only of the empirical fields, and the nonhydrodynamic terms are of order ε^2.

To single out the nonhydrodynamic contribution to F_ε let us decompose F_ε into a part which is Gibbsian with parameters slowly depending on the microscopic variables and depending on ε by means of a series in ε, and a remainder. More explicitly, we put

$$F_\varepsilon = G_\varepsilon + \varepsilon^2 G_0 R_\varepsilon \tag{2.27}$$

with

$$G_\varepsilon = Z_\varepsilon^{-1} \exp\left\{\sum_{i,\mu} \lambda_\varepsilon^\mu(x_i, t)\, z_i^\mu\right\}; \qquad \lambda_\varepsilon^\mu(x, t) = \sum_{n=0}^{\infty} \varepsilon^n \lambda_n^\mu(x, t); \qquad \lambda_0^\mu = \text{const} \tag{2.28}$$

G_0 is the zeroth-order term in the expansion, and so, as explained before, is the global equilibrium. We include all the hydrodynamic terms in G_ε and we can assume that in R_ε there are no terms which are combinations of the invariant quantities z^α with coefficients depending on the macroscopic variables, since these terms are already present in G_ε. In other words, we put

$$\hat{R}_\varepsilon = 0 \tag{2.29}$$

We also assume that for any $t > 0$

$$R_\varepsilon(t) = R(t) + O(\varepsilon) \tag{2.30}$$

We need an explicit expression for R in terms of the empirical fields, so that, inserting (2.27) in the conservation laws averaged with respect to F_ε, we can get closed equations for the empirical fields up to order ε. To find such an expression, we insert the expansion (2.27) for F_ε in the Liouville equation (2.26) and integrate on time

$$[G_\varepsilon(t) - G_\varepsilon(0)] + \varepsilon^2 G_0[R_\varepsilon(t) - R_\varepsilon(0)] = \int_0^t [\varepsilon^{-2}\mathscr{L}^* G_\varepsilon + \mathscr{L}^* G_0 R_\varepsilon] \tag{2.31}$$

The LHS of (2.31) goes to 0 in the limit $\varepsilon \to 0$, since the only term of order 1 is constant in time due to the assumptions on λ_0. Hence we have

$$\int_0^t \{\varepsilon^{-1}\mathscr{L}^* g_1 + \mathscr{L}^* h\} = -\int_0^t \mathscr{L}^* R + O(\varepsilon) \tag{2.32}$$

where $g_1 = \sum_{j,\mu} \lambda_1^\mu(x_j, t)\, z_j^\mu$ and we have used

$$G_\varepsilon = G_0\{1 + \varepsilon[g_1 - \langle g_1 \rangle] + \varepsilon^2 h\} + O(\varepsilon^3) \tag{2.33}$$

Here h is a function of the invariant fields and $\langle \cdot \rangle$ is the average w.r.t. G_0.

Since g_1 is a linear combination of the invariant quantities z with coefficients depending on the macroscopic variables, the action of $\mathscr{L}^*$ on it gives a linear combination of the currents w with a factor ε. Therefore the first term in the LHS of (2.32) is of order 1. Moreover, for the same reasons the second term goes to zero. In conclusion, R satisfies the equation

$$\int_0^t \left[\mathscr{L}^* R - \sum_i \sum_{\mu,\gamma} \frac{\partial \lambda_1^\mu}{\partial x_i^\gamma} (x_i, s)(1 - \mathscr{P})\, w_i^{\mu\gamma} \right] = 0 \tag{2.34}$$

We assume that there exists a unique solution $R(t)$ to (2.34) such that $\hat{R}(t) = 0$, which we write formally as

$$\mathscr{L}^{*-1} \sum_i \sum_{\mu,\gamma} \frac{\partial \lambda_1^\mu}{\partial x_i^\gamma} (x_i, t)(1 - \mathscr{P})\, w_i^{\mu\gamma}$$

This is the assumption we really need on the inverse of $\mathscr{L}^*$ to get the result. In the stochastic model of Section 4 the nongradient method provides a way to construct such a solution.

3. INCOMPRESSIBLE NAVIER–STOKES EQUATIONS

The incompressible limit corresponds to the assumption that the velocity field is small compared with the sound speed. In other words, we assume that $U^\mu(x, t) \equiv \langle z^\mu(x) \rangle_{F_\varepsilon(t)}$, $\mu = 1,..., d$, starts with a term of order ε. Under the assumptions on F_ε, this corresponds to choosing $\lambda_0^\mu = 0$ for $\mu = 1,..., d$. On the other hand, one finds $U^\mu(x, t) = \frac{1}{2} \varepsilon \rho T \lambda_1^\mu(x, t) + O(\varepsilon^2)$ with T, the constant temperature of the Gibbs state G_0, given by $(\lambda_0^{d+1})^{-1}$ and ρ the constant density of the Gibbs state G_0 corresponding to the constant chemical potential λ_0^0. We denote by $u^\mu(x, t)$ the rescaled velocity field given by $u^\mu(x, t) = \frac{1}{2} T \lambda_1^\mu(x, t)$. In this situation the continuity equation reduces to the incompressibility condition $\operatorname{div} u = 0$. To obtain it, we start from the conservation law for the empirical density (2.7) and we take the expectation with respect to the nonequilibrium measure $F_\varepsilon(t)$,

$$\begin{aligned} &\left\langle \varepsilon^d \sum_i f(x_i) \right\rangle_{F_\varepsilon(t)} - \left\langle \varepsilon^d \sum_i f(x_i) \right\rangle_{F_\varepsilon(0)} \\ &\quad = \varepsilon^{-1} \int_0^t ds \left\langle \varepsilon^d \sum_{i,k} \frac{\partial f}{\partial x_i^k} (x_i)\, v_i^k \right\rangle_{F_\varepsilon(s)} \end{aligned} \tag{3.1}$$

Using (2.27), (2.28), and (2.30), we see that the LHS of (3.1) goes to zero and (3.1) becomes

$$\int_0^t ds \left\langle g_1 \sum_i \sum_{k=1}^d \frac{\partial f}{\partial x_i^k}(x_i)\, v_i^k \right\rangle$$
$$= \int_0^t ds \left\langle \sum_j \sum_{\mu=0}^{d+1} \lambda_1^\mu(x_j, t)\, z_j^\mu \varepsilon^d \sum_i \sum_{k=1}^d \frac{\partial f}{\partial x_i^k}(x_i)\, v_i^k \right\rangle = O(\varepsilon) \quad (3.2)$$

where $\langle \cdot \rangle$ is the average with respect the Gibbs measure G_0. Since G_0 is Gaussian in the velocities, the terms with $\mu = 0$, $d+1$ do not contribute (they are averages of odd polynomials in v) and the LHS of (3.2) becomes

$$\left\langle \varepsilon^d \sum_i \sum_{\mu, k=1}^d \lambda_1^\mu(x_i, t) \frac{\partial f}{\partial x_i^\mu}(x_i)\, v_i^\mu v_i^k \right\rangle \quad (3.3)$$

Since the average on G_0 of $v_i^\mu v_i^k$ contributes only for $k = \mu$, in the limit $\varepsilon \to 0$ we have

$$\int_0^t ds \int dx \sum_{\mu=1}^d \frac{\partial(\rho u^\mu)}{\partial x^\mu}(x, t)\, f(x) = 0 \quad (3.4)$$

for any test function f and for any t. Hence

$$\operatorname{div} u = 0 \quad (3.5)$$

We examine now the second conservation law (2.8). By averaging as before, for $\beta = 1, \dots, d$, we get

$$\left\langle \varepsilon^d \sum_i f(x_i)\, v_i^\beta \right\rangle_{F_\varepsilon(t)} - \left\langle \varepsilon^d \sum_i f(x_i)\, v_i^\beta \right\rangle_{F_\varepsilon(0)}$$
$$= \varepsilon^{-1} \int_0^t ds \left\langle \varepsilon^d \sum_i \sum_{k=1}^d \frac{\partial f}{\partial x_i^k}(x_i) \left\{ v_i^k v_i^\beta + \frac{1}{2} \sum_{j \neq i} \Psi^{\beta k}(\varepsilon^{-1}(x_i - x_j)) \right\} \right\rangle_{F_\varepsilon(s)}$$
$$+ O(\varepsilon) \quad (3.6)$$

Using the assumptions on $F_\varepsilon(t)$ we see that the LHS is of order ε, since the term of order 1 vanishes because λ_0^μ are constant. It is convenient to rewrite the integrand in the RHS of (3.6) as

$$\int dx \frac{\partial f}{\partial x^k}(x) \langle \varepsilon^{-1} w^{\beta k}(x) \rangle_{F_\varepsilon} \quad (3.7)$$

Using the assumptions (2.27)–(2.30), we have

$$\varepsilon^{-1} \langle w^{\beta k} \rangle_{F_\varepsilon} = \varepsilon^{-1} \langle w^{\beta k} \rangle_{G_\varepsilon} + \varepsilon \langle w^{\beta k} R_\varepsilon \rangle \quad (3.8)$$

We introduce the currents $\tilde{w}^{\beta k}$ as given by the expression (2.16) with the velocities v_i replaced by $\tilde{v}_i = v_i - \varepsilon u(x_i)$. Then

$$w_i^{\beta k} = \tilde{w}_i^{\beta k} + \varepsilon^2 u^k(x_i)\, u^\beta(x_i) + \varepsilon u^k(x_i)\, \tilde{v}_i^\beta + \varepsilon u^\beta(x_i)\, \tilde{v}_i^k \tag{3.9}$$

For the symmetry of the measure G_ε we have $\langle \tilde{w}^{\beta k}(x)\rangle_{G_\varepsilon} = O(\varepsilon^4)$ if $k \neq \beta$. The average of $\tilde{w}^{\beta\beta}$, $\beta = 1,..., d$, with respect the local Gibbs state G_ε is, by the virial theorem, the thermodynamic pressure P_ε in the state G_ε.[21] Therefore Eq. (3.6) and assumption (2.30) imply

$$\varepsilon^{-1} \int dx\, \nabla f(x)\, P^\varepsilon(x, s) = O(\varepsilon) \tag{3.10}$$

Since P^ε is a function of the thermodynamic parameters λ_ε, we can expand it in series of ε as $\sum_k \varepsilon^k P_k$, where $P_k = 1/k!\, (d^k P^\varepsilon / d\varepsilon^k)|_{\varepsilon=0}$. We have that P_0 is constant since it is a function of the constants λ_0^0 and λ_0^{d+1}, while

$$P_1 = \sum_{\mu=0}^{d+1} \left.\frac{\partial P^\varepsilon}{\partial \lambda_\varepsilon^\mu}\right|_{\varepsilon=0} \lambda_1^\mu$$

In order to fulfill (3.10) for any test function f, P_1 has to be constant.

To determine the equation for $u^\mu(x, t)$, we have to rescale the empirical velocity field. This means that we have to look at the empirical field

$$z^\alpha(x) = \varepsilon^{-1} \varepsilon^d \sum_i v_i^\alpha \delta(x_i - x), \qquad \alpha = 1,..., d$$

We proceed as we did before to obtain (3.6), but we have to look at the explicit form of the term $O(\varepsilon)$ because it has to be divided by ε. We have

$$\begin{aligned}
&\left\langle \varepsilon^{d-1} \sum_i f(x_i)\, v_i^\beta \right\rangle_{F_\varepsilon(t)} - \left\langle \varepsilon^{d-1} \sum_i f(x_i)\, v_i^\beta \right\rangle_{F_\varepsilon(0)} \\
&\quad = \varepsilon^{-2} \int_0^t ds \left\langle \varepsilon^d \sum_i \sum_{k=1}^d \frac{\partial f}{\partial x_i^k}(x_i) \left\{ v_i^k v_i^\beta + \frac{1}{2} \sum_{j \neq i} \Psi^{\beta k}(\varepsilon^{-1}(x_i - x_j)) \right\} \right\rangle_{F_\varepsilon(s)} \\
&\qquad + \int_0^t ds \left\langle \frac{1}{2} \varepsilon^d \sum_{i \neq j} \nabla_\beta V(\varepsilon^{-1}(x_i - x_j))\, D(x_i - x_j) \right\rangle_{F_\varepsilon(s)} + O(\varepsilon)
\end{aligned} \tag{3.11}$$

This equation, to the lowest order in ε, reduces to the Navier–Stokes equation for the velocity field u. The argument is the following. The LHS of (3.11) is given by

$$\int dx\, f(x)\, \rho[u(x, t) - u(x, 0)] \tag{3.12}$$

up to terms of order ε.

To get the INS, we have to compute again the nonequilibrium average of the velocity current tensor $w^{\beta k}$, but now there is a factor ε^{-2} in front of it. Therefore we see that in this case also the terms of order ε^2 in (2.27) have to be taken into account. First of all we observe that the term containing D goes to 0 as $\varepsilon \to 0$ because the lowest order is given by an average with respect to G_0 (Gibbs measure with constant parameters), hence it vanishes because it contains derivatives of f. The other terms are at least of order ε, hence do not contribute to the lowest order.

Moreover, we have

$$\begin{aligned}
&\varepsilon^{-2}\langle w^{\beta k}(x)\rangle_{G_\varepsilon} + \langle w^{\beta k}(x)\, R_\varepsilon\rangle \\
&\quad = \varepsilon^{-2}\langle \tilde{w}^{\beta k}\rangle_{G_\varepsilon} + \rho u^\beta u^k + \langle w^{\beta k} R_\varepsilon\rangle + O(\varepsilon) \\
&\quad = \varepsilon^{-2} P_0 + \varepsilon^{-1} P_1 + P_2 + \rho u^\beta u^k + \langle R w^{\beta k}(x)\rangle + O(\varepsilon) \qquad (3.13)
\end{aligned}$$

The first two terms of (3.13) do not contribute to the INS, because P_0 and P_1 are constant. The fourth term in (3.13) gives the nonlinear transport term, while P_2 represents the second-order correction to the thermodynamic pressure P_ε and gives rise to the unknown pressure p appearing in the INS.

The last term in the RHS of (3.13) is determined by R, the nonequilibrium part of the distribution F_ε, which takes into account the fast modes in the system, namely the nonconserved quantities. They appear at the hydrodynamic level only through dissipative effects and determine the expression of the transport coefficients.

To compute it, let us first introduce $\bar{w}^{\beta k} = \tilde{w}^{\beta k} - \mathscr{P}\tilde{w}^{\beta k}$ and notice that (2.29) implies

$$\langle R\mathscr{P} w^{\alpha\beta}(x)\rangle = 0 \qquad (3.14)$$

Again by (2.29) we can use the "identity" $(\mathscr{L}^*)^{-1}\,\mathscr{L}^* R = R$ and (2.34) to get [recall the notation introduced after (2.17)]

$$\begin{aligned}
&\int dx \sum_{k=1}^{d} \frac{\partial f}{\partial x^k} \langle \bar{w}^{\beta k}(x)(\mathscr{L}^*)^{-1}\,\mathscr{L}^* R\rangle \\
&\quad = \left\langle \mathscr{L}^* R \mathscr{L}^{-1} \varepsilon^d \sum_i \sum_{k=1}^{d} \bar{w}_i^{\beta k} \frac{\partial f}{\partial x_i^k}(x_i)\right\rangle \\
&\quad = \left\langle \sum_j \sum_{\mu=0}^{d+1} \sum_{l,k=1}^{d} \frac{\partial \lambda_1^\mu}{\partial x_j^l}(x_j, s)(1-\mathscr{P})\, w_j^{\mu l} \mathscr{L}^{-1} \varepsilon^d \sum_i \bar{w}_i^{\beta k} \frac{\partial f}{\partial x_i^k}(x_i)\right\rangle \qquad (3.15)
\end{aligned}$$

We remark that $\mathscr{L}^{-1}$ is "well defined" on $\bar{w}$ by the assumptions discussed in Section 2.

To find the expression of the transport coefficient, we consider

$$\left\langle \sum_j \frac{\partial \lambda_1^\mu}{\partial x_j^l}(x_j, s)(1-\mathscr{P})\, w_j^{\mu l} \mathscr{L}^{-1} \varepsilon^d \sum_i \bar{w}_i^{\beta k} \frac{\partial f}{\partial x_i^k}(x_i) \right\rangle$$

$$= \int dy \frac{\partial \lambda_1^\mu}{\partial y^l}(y, s) \int \varepsilon^{-d}\, dz \frac{\partial f}{\partial z^k}(z) \langle \bar{w}^{\mu l}(x)\, \mathscr{L}^{-1} \bar{w}^{\beta k}(z) \rangle \quad (3.16)$$

Since the Gibbsian state G_0 is invariant under translations on $\mathbb{R}^d$, we have

$$\text{RHS of (3.16)} = \int dy \frac{\partial \lambda_1^\mu}{\partial y^l}(y, s) \int d\xi \frac{\partial f}{\partial y^k}(y + \varepsilon\xi) \langle \bar{w}^{\mu l}(0)\, \mathscr{L}^{-1} \bar{w}^{\beta k}(\xi) \rangle \quad (3.17)$$

where we have changed the variable z in $\xi = \varepsilon^{-1}(z - y)$ absorbing the factor ε^{-d}. Hence, provided that $\langle \bar{w}^{\mu l}(0)\, \mathscr{L}^{-1} \bar{w}^{\beta k}(\xi) \rangle$ decays fast enough for large ξ, up to $O(\varepsilon)$ we have

$$\text{RHS of (3.16)} = \int dy \frac{\partial \lambda_1^\mu}{\partial y^l}(y, s) \frac{\partial f}{\partial y^k}(y) \int d\xi\, \langle \bar{w}^{\mu l}(0)\, \mathscr{L}^{-1} \bar{w}^{\beta k}(\xi) \rangle \quad (3.18)$$

To conclude the argument, we transform $\mathscr{L}^{-1}$ in a time integral of $\exp(t\mathscr{L})$:

$$-\langle \bar{w}^{\mu l}(0)\, \mathscr{L}^{-1} \bar{w}^{\beta k}(\xi) \rangle = \int_0^\infty d\tau\, \langle \bar{w}^{\mu l}(\xi, \tau)\, \bar{w}^{\beta k}(0, 0) \rangle \quad (3.19)$$

The symmetries of the microscopic current–current correlations imply[4] that the correlations for $\mu = 0,\ d+1$ vanish and

$$\int d\xi\, \langle \bar{w}^{\mu l}(\xi, \tau)\, \bar{w}^{\beta k}(0, 0) \rangle = c(\tau)[\delta_{kl}\delta_{\beta\mu} + \delta_{k\mu}\delta_{\beta l}] + c'(\tau)\, \delta_{\beta k}\delta_{l\mu} \quad (3.20)$$

Therefore the time integral of (3.20) has only two independent coefficients,

$$\int_0^\infty d\tau\, c(\tau) = 2\eta T; \qquad \int_0^\infty d\tau\, c'(\tau) = 2T\left(\zeta - \frac{2}{d}\eta\right) \quad (3.21)$$

where η and ζ are the shear viscosity and the bulk viscosity, respectively. They are finite if the correlations decay sufficiently fast to make the time integrals in (3.21) convergent. Notice that the subtraction of $\mathscr{P}\bar{w}^{\alpha\beta}$ has been crucial, because the self-correlation of the slow part of the current does not decay in time.

Since we already know that div $u=0$, the term proportional to the bulk viscosity does not appear in the limiting equation. Putting all the terms together, we have the following equation to the lowest order in ε:

$$\int dx\, f(x)\, \rho[u^{\beta}(x,t)-u^{\beta}(x,0)]$$

$$=\int_0^t ds \int dy \sum_{k=1}^{d} \frac{\partial f}{\partial y^k}(y)\left\{\rho u^{\beta}(y,s)\, u^k(y,s)-\eta\,\frac{\partial u^{\beta}}{\partial y^k}(y,s)+p(y,s)\right\} \tag{3.22}$$

for any test function f, and hence the incompressible Navier–Stokes equation. The viscosity is given by

$$\eta=\frac{1}{2T}\int_0^{\infty} d\tau \int d\xi\, \langle \bar{w}^{12}(\xi,\tau)\, \bar{w}^{12}(0,0)\rangle \tag{3.23}$$

and is independent of the space coordinates because the current–current correlation involved is computed at the global equilibrium.

The computation gives also the Green–Kubo formula for the bulk viscosity ζ,

$$\zeta=\frac{1}{2d^2T}\int_0^{\infty} d\tau \int d\xi \left[\left\langle \sum_{\alpha} \bar{w}^{\alpha\alpha}(\xi,\tau) \sum_{\gamma} \bar{w}^{\gamma\gamma}(0,0)\right\rangle-\left\langle \sum_{\alpha} \bar{w}^{\alpha\alpha}\right\rangle\left\langle \sum_{\gamma} \bar{w}^{\gamma\gamma}\right\rangle\right] \tag{3.24}$$

The usual expression given in ref. 4 is recovered using the explicit form of the projector $\mathscr{P}$.

By means of the arguments developed before it is possible to find an equation for the first correction to the kinetic energy. We will use below the following remark on the conservation of the mass. Since the density current is a locally invariant field, taking into account (2.29), it follows that the nonequilibrium average of (2.15) will be determined only by the Gibbsian part G_{ε}. Thus the equation for the averages is

$$\int dx\, f(x)[\rho_{\varepsilon}(x,t)-\rho_{\varepsilon}(x,0)]=-\int_0^t \varepsilon^{-1} \int dx\, \mathrm{div}(\rho_{\varepsilon} u_{\varepsilon})\, f(x) \tag{3.25}$$

where

$$\rho_{\varepsilon}=\langle z^0\rangle_{G_{\varepsilon}} \qquad \text{and} \qquad \rho_{\varepsilon} u_{\varepsilon}^{\mu}=\langle z^{\mu}\rangle_{G_{\varepsilon}}, \quad \mu=1,\dots,d$$

Now we consider the conservation law for the energy (2.9). We are interested in the first correction to the energy, since at zeroth order the energy is a constant, which we call e_0. Therefore we look for the equation for the quantity $\varepsilon^{-1}(z_i^{d+1} - e_0)$. By (2.9) we have

$$\frac{d}{dt}\varepsilon^{-1}\varepsilon^d \sum_i f(x_i)(z_i^{d+1} - e_0) = \varepsilon^{-2}\varepsilon^d \sum_i \sum_{k=1}^{d} \frac{\partial f}{\partial x^k}(x_i)\, w_i^{d+1,k} + O(\varepsilon) \quad (3.26)$$

We only sketch the argument to get the limiting equation, since the procedure is the same as in the previous cases. We need to evaluate $\langle w^{d+1,k}(x)\, R_\varepsilon\rangle$ and $\varepsilon^{-2}\langle w^{d+1,k}(x)\rangle_{G_\varepsilon}$.

The first of them gives the diffusive correction. We introduce $\tilde{w}^{d+1,k}$ and $\tilde{z}^{d+1}$ defined by (2.17) and (2.4) with v replaced by $\tilde{v}$. We have

$$w_i^{d+1,k} = \tilde{w}_i^{d+1,k} + \varepsilon\left\{u^k(x_i)\, z_i^{d+1}\right.$$

$$\left. + \sum_\gamma \left(u^\gamma \tilde{v}_i^\gamma \tilde{v}_i^k + \Psi^{\gamma k}(\varepsilon^{-1}|x_i - x_j|)\, \tfrac{1}{2}[u^\gamma(x_i) + u^\gamma(x_j)]\right)\right\} \quad (3.27)$$

Then, as before,

$$\langle \bar{w}^{d+1,k} R_\varepsilon\rangle = \left\langle \sum_j \sum_{l=1}^{d} \sum_{\mu=0}^{d+1} \frac{\partial \lambda_1^\mu}{\partial x_j^l}(x_j)\, \bar{w}_j^{\mu l} \mathscr{L}^{-1} \bar{w}^{d+1,k}(x)\right\rangle + O(\varepsilon) \quad (3.28)$$

where $\bar{w}^{d+1,k} = \tilde{w}^{d+1,k} - \mathscr{P}(\tilde{w}^{d+1,k})$. Because of time-reversal and rotation invariance of the Gibbs state, the only correlations different from zero are[4]

$$\int dx\, \langle \bar{w}^{d+1,k}(x,\tau)\, \bar{w}^{d+1,l}(0,0)\rangle = \delta_{lk}\, a(\tau) \quad (3.29)$$

and $\int d\tau\, a(\tau) = 2\kappa T^2$. Therefore the conductivity κ is given by

$$\kappa = \frac{1}{2dT^2}\int d\tau \left\{\int d\xi \left\langle \sum_{k,l} w^{d+1,k}(x,\tau)\, w^{d+1,l}(0,0)\right\rangle - d\left(\frac{T(e+P)^2}{\rho}\right)\right\} \quad (3.30)$$

We observe that, since $\lambda_\varepsilon = -(T_\varepsilon)^{-1}$, λ_1^{d+1} is given by $T_1(T^{-2})$.

Using the previous arguments, one can see that the second term in (3.27) gives no contribution to $\langle w^{d+1,k} R_\varepsilon\rangle$ in the limit $\varepsilon \to 0$.

The mean of the energy current on the Gibbs state G_ε, i.e., $\langle w^{d+1,k}\rangle_{G_\varepsilon}$, is nothing but $(\rho_\varepsilon e_\varepsilon + P_\varepsilon)\, u_\varepsilon$,[4] where $\rho_\varepsilon e_\varepsilon = \langle z^{d+1}\rangle_{G_\varepsilon}$.

Summarizing, if we take the average of (3.26) with respect F_ε, we obtain

$$\varepsilon^{-1} \int dx\, f(x)[\rho_\varepsilon e_\varepsilon(x, t) - \rho_\varepsilon e_\varepsilon(x, 0)]$$

$$= \int_0^t ds \int dx\, f(x)\{-\varepsilon^{-2} \operatorname{div}[(\rho_\varepsilon e_\varepsilon + P_\varepsilon)\, u_\varepsilon](x, s) + k\, \Delta T_1(x, s)\} + O(\varepsilon) \tag{3.31}$$

for any test function f. Hence, using (3.25),

$$\varepsilon^{-1}\rho_\varepsilon \partial_t e_\varepsilon + \varepsilon^{-2} u_\varepsilon \rho_\varepsilon \nabla e_\varepsilon + \varepsilon^{-2} \operatorname{div}(P_\varepsilon u_\varepsilon) = \kappa\, \Delta T_1 + O(\varepsilon) \tag{3.32}$$

after switching to the differential form of the equation for the sake of simplicity. Writing e_ε, P_ε, and u_ε as series in ε with coefficients e_n, P_n, and u_n (recall $u_0 = 0$, $u_1 = u$), we have, using $\nabla P_1 = 0$ and $\operatorname{div} u = 0$

$$\text{LHS of (3.32)} = \rho \partial_t e_1 + P_0 \operatorname{div} u_2 + \rho u \cdot \nabla e_1 + O(\varepsilon) \tag{3.33}$$

As the next step we eliminate $\operatorname{div} u_2$ by means of Eq. (3.25), which we rewrite in the form

$$D_t \rho_1 + \rho \operatorname{div} u_2 = O(\varepsilon) \tag{3.34}$$

where $D_t f \equiv \partial_t f + u \cdot \nabla f$. We get

$$\rho D_t e_1 - \rho^{-1} P_0 D_t \rho_1 = \kappa\, \Delta T_1 + O(\varepsilon) \tag{3.35}$$

Let us note that only the internal energy contributes to e_1, because the kinetic energy is of order ε^2. Hence e_1 can be written as a function of ρ_1 and T_1 and we have

$$\left(\rho \frac{\partial e_1}{\partial \rho_1} - \frac{P_0}{\rho}\right) D_t \rho_1 + \frac{\partial e_1}{\partial T_1} D_t T_1 = \kappa\, \Delta T_1 + O(\varepsilon) \tag{3.36}$$

The Boussinesq condition, stating that P_1 is constant, implies

$$\left.\frac{\partial P_\varepsilon}{\partial \rho}\right|_0 D_t \rho_1 + \left.\frac{\partial P_\varepsilon}{\partial T}\right|_0 D_t T_1 = 0 \tag{3.37}$$

We can eliminate $D_t \rho_1$ using above relation and, up to $O(\varepsilon)$, we get

$$c(\partial_t T_1 + u \cdot T_1) = \kappa\, \Delta T_1 \tag{3.38}$$

where c is given by ρ times the specific heat at constant pressure. Equation (3.38) for $u = 0$ is the Fourier law.

4. ASYMMETRIC SIMPLE EXCLUSION PROCESS

In this section we briefly review the result in ref. 14 because it supports our previous considerations in the sense that in a situation where the analysis can be made rigorous, the results confirm the arguments presented here.

In fact we consider a lattice gas with hard-core exclusion as a simple example of a stochastic system of particles for which it is possible to prove rigorously a sort of INS limit. The model is as follows: we consider a system of particles on a d-dimensional lattice $\mathbb{Z}^d$, with periodic boundary conditions on a cubic region of size ε^{-1} which we denote by $\mathbb{T}_\varepsilon$. The particles jump independently with intensity $p_{x,y} \geqslant 0$ from the site x to the site y if it is empty. We denote by $\eta(x) = 0, 1$ the occupation number per site; η is a configuration of the system and the configuration space is $\{0, 1\}^{\mathbb{T}_\varepsilon}$.

The generator of the stochastic dynamics is

$$\mathscr{L}f = \sum_{xy} c(x, y, \eta)[f(\eta^{xy}) - f(\eta)] \tag{4.1}$$

with

$$c(x, y, \eta) = p_{x,y}\eta(x)[1 - \eta(y)] \tag{4.2}$$

and η^{xy} is the configuration in which $\eta(x)$ and $\eta(y)$ are exchanged. We restrict ourselves to the nearest neighbor case, i.e., we assume that $p_{x,y}$ are nonvanishing if and only if $|y - x| = 1$. We denote by e the unit vectors on the lattice with nonnegative components and put $p_e = p_{x,y}$ if $y = x + e$ and $p_{-e} = p_{y,x}$. It is also convenient to fix $p_e + p_{-e} = 2$ for all e.

This system has only one locally conserved field, the density. An invariant measure for the dynamics is the product measure $Z_\lambda^{-1} \prod_{x \in \mathbb{T}_\varepsilon} \exp \lambda\eta(x)$.

On the Euler time scale, for any smooth function f on $\mathscr{R}^d$, define

$$r_t(f) = \varepsilon^d \sum_{x \in \mathbb{T}_\varepsilon} f(\varepsilon x)\, \eta_{\varepsilon^{-1}t}(x) \tag{4.3}$$

where $\eta_t(x)$ denotes the number of particles in x at time t. It has been proven in ref. 16 that $r_t(f)$ converges, as $\varepsilon \to 0$, to a limit given by $\int dz\, f(z)\, \rho(z, t)$, with $\rho(z, t)$ a solution of the d-dimensional nonviscous Burgers equation

$$\partial_t \rho + F \cdot \nabla[\rho(1 - \rho)] = 0 \tag{4.4}$$

where F is the driving field given by $F_e = p_e - p_{-e}$.

To see diffusive effects, as usual we have to wait for microscopic times of order ε^{-2}. The analysis of the corrections of order ε to (4.4) suggests that the macroscopic equation is the viscous Burgers equation. On the other hand, such an equation is not invariant under the diffusive scaling. The situation is quite similar to the one we described for the Navier–Stokes equation, but in a simpler case with only one conserved quantity. In fact the viscous Burgers equation is invariant if in addition to the diffusive scaling for space and time we consider perturbations of order ε to the constant-density profile. Let $0 < \theta < 1$ be a constant density and assume $\rho = \theta - \varepsilon u$. Then, on times $\varepsilon^{-2}t$ we expect for u an equation of the form

$$\frac{\partial u}{\partial t} + \varepsilon^{-1} v \cdot \nabla u + F \cdot \nabla u^2 = \sum_{i,j=1}^{d} D_{i,j} \frac{\partial^2 u}{\partial z_i \, \partial z_j} \tag{4.5}$$

with $v = (1 - 2\theta) F$ and $D_{i,j}$ some diffusion matrix. We remove the diverging term by considering, instead of u, $m(z, t) = u(z + \varepsilon^{-1} vt, t)$, which satisfies

$$\frac{\partial m}{\partial t} + F \cdot \nabla m^2 = \sum_{i,j=1}^{d} D_{i,j} \frac{\partial^2 m}{\partial z_i \, \partial z_j} \tag{4.6}$$

The above considerations suggest that we introduce the rescaled empirical field defined for any test function f as

$$z_t^\varepsilon(f) = \varepsilon^{d-1} \sum_{x \in \mathbb{T}_\varepsilon} f(\varepsilon x + \varepsilon^{-1} vt)[\theta - \eta_{\varepsilon^{-2}t}(x)]$$

In ref. 14, using the "nongradient" method,[17] the entropy method,[18,3] and a multiscale analysis, the following theorem is proved:

Theorem. Let $\eta(t)$ be the stochastic process described before. Moreover, let $d \geqslant 3$. We choose the sequence of initial measures as

$$\mu^\varepsilon = Z_\varepsilon^{-1} \exp \sum_{x \in \mathbb{T}_\varepsilon} \{[\beta + \varepsilon \lambda_0(\varepsilon x)] \, \eta(x)\}$$

and let $m_0(z)$ be such that for any $\delta > 0$ and any f smooth and of compact support

$$\lim_{\varepsilon \to 0} \operatorname{Prob} \left\{ \left| z_0^\varepsilon(f) - \int dz \, f(z) \, m_0(z) \right| > \delta \right\} = 0$$

Then there is a symmetric matrix D satisfying

$$D > 1 \tag{4.7}$$

(as a matrix), such that for any $t \geqslant 0$ and $\delta > 0$

$$\lim_{\varepsilon \to 0} \text{Prob} \left\{ \left| z_t^\varepsilon(f) - \int dx\, f(x)\, u(z, t) \right| > \delta \right\} = 0$$

where $m(z, t)$ is the unique smooth solution of the d-dimensional nonlinear diffusion equation (4.6) (the viscous Burgers equation) with initial condition $m_0(z)$.

Remark 1. The result of the theorem is expressed as a law of large numbers, but with an extra factor ε^{-1}. Hence, as remarked before, it cannot hold in dimension less than 3, because the fluctuations are too big. Actually this is not the only reason for the restriction; in fact the multiscale analysis on which it is based fails in dimension less than 3. This is in agreement with the conjecture[4] that the diffusion coefficient for the ASEP is infinite in dimensions 1 and 2.

Remark 2. Consider the symmetric exclusion process with $\bar{p}_e = \bar{p}_{-e} = \frac{1}{2}(p_e + p_{-e}) = 1$. The limit satisfies the diffusion equation with diffusion matrix 1. Hence the inequality (4.7) shows that there is a contribution to the diffusion coming from the asymmetry of the jumps. Since such an asymmetry may be interpreted as a "deterministic" motion added to the symmetric diffusion, its contribution to the diffusion is the analog of the viscosity for the Hamiltonian systems, where, of course, there is no symmetric part. More explicitly, the expression we get for D in ref. 14 is equivalent to the heuristic Green–Kubo formula for D given in ref. 19,

$$D_{ee'} = \delta_{ee'} + (2\chi)^{-1} \int_0^\infty dt \sum_x \left[\langle \sigma_e \partial_{0e} e^{\mathcal{L}t} \sigma_{x,x+e'} \rangle \right] \tag{4.8}$$

where $\partial_{0e} f(\eta) = f(\eta^{0e})$. The first term is the contribution coming from the stochastic motion and the second one is a time-integrated current–current correlation. We do not know, even in this case, how to give sense to the integral in (4.8) or equivalently to $\mathcal{L}^{-1}$, but we use properties of the symmetric part of the generator (spectral gap, logarithmic Sobolev inequality) to obtain a variational formula for D which provides a rigorous version of the Green–Kubo formula.

Remark 3. The strict analog of the incompressible limit is actually the case $\theta = 1/2$, for which $v = 0$ and the "velocity" is of order ε. The case considered in ref. 14 is slightly more general because of the diverging transport term in the limit equation (4.5). It has been possible to manage it (because v is constant in space) by considering a frame of reference moving with speed $\varepsilon^{-1}v$. Hence this is an example (indeed very simple) in which one can give sense to the Navier–Stokes correction also in the presence of a diverging Euler contribution.

Remark 4. The above theorem is proved showing that the nonequilibrium measure is close, in the sense of the relative entropy, to the local equilibrium with parameter of order ε (up to a constant) plus a nonhydrodynamic correction of order ε^2, whose form is determined by the nongradient method and vanishing only in the symmetric case. This result is the counterpart of assumptions (2.27)–(2.30) and (2.34).

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